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# Topology optimization for enhanced dynamic fracture resistance of structures

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## Abstract

A topology optimization framework for improving the dynamic fracture resistance of structures is proposed. The phase field method for fracture is combined with Solid Isotropic Material with Penalization (SIMP) topology optimization. The topology optimization problem is defined as minimizing the fracture energy during the whole dynamic loading process, from initiation of cracks to full failure of the structure, under volume and compliance constraints. Semi analytical expressions of sensitivities in a dynamic context are provided to solve the topology optimization problem efficiently. Numerical examples involving structures subjected to impact loading are investigated. It is shown that the present framework allows a significant reduction of the fracture energy as compared to designs obtained by static optimization.

*Keywords:* Topology optimization, Phase-field method, Dynamics, Structural optimization, SIMP, Fracture

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## 1. Introduction

Since the late 1980s [1], topology optimization (TO) has evolved into one of the most powerful numerical design methods. In a given design domain, topology optimization approaches [2, 3, 4, 5] generate the optimal topological designs that minimize or maximize an objective function with certain design constraints. The problems solved by TO have gradually evolved from simple linear global optimization problems to localized stress problem [6], to dynamic problems [7], nonlinear problems [8], uncertainties [9], multidisciplinary integration [10] or towards large-scale problem [11], among many others. These achievements have demonstrated the potential of TO in both academic and industrial applications.

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10 Recently, an exciting new branch of TO has emerged to enhance the mechanical resistance  
11 to damage or cracks in structures and materials. The pioneering work on this topic might be  
12 traced back to Challis et al. [12], in which a level-set method was proposed to maximize the  
13 structural fracture resistance. In [13], Kang et al. used a  $J$ -integral approach to predict crack  
14 opening at predefined locations. In [14, 15], non-local damage field was considered for the first  
15 time in TO to obtain the optimal mechanical resistance design of concrete structures and their  
16 reinforcement. Similarly, Kato and Ramm [16] investigated fiber-reinforced composites considering  
17 a damage model, in which the layout of the multi-phase materials was optimized. James and  
18 Waisman [17] developed a non-local damage-TO- coupled algorithm for failure reduction, in which  
19 the maximal damage was constrained, similarly to what is usually done in stress-constrained  
20 problems. In [18, 19], Li et al. investigated TO methods involving stored energy while constraining  
21 the elastoplastic-damage. More recently, Russ and Waisman [20] proposed a method for the  
22 structural resistance of both ductile failure and buckling in a new aggregated optimization objective  
23 with local ductile failure constraints. Liu et al. [21] investigated multi-material fracture resistance  
24 TO including cohesive models.

25 Note that the above mentioned studies do not include a complete damage or crack evolution  
26 involving the whole loading history. In [22], Zhang et al. firstly included a full crack propagation  
27 analysis within TO for fracture resistance designs, using the X-FEM [23, 24] method. However,  
28 the complexity of XFEM for dealing with initiation and complex cracks configurations strongly  
29 restricts its use within TO analysis. In [25], Xia et al. combined for the first time TO with frac-  
30 ture phase field analysis to maximize the resistance of composite structures. The variational phase  
31 field approach to fracture [26, 27, 28, 29, 30, 31] has unique advantages to deal with initiation,  
32 propagation of multiple, complex, 3D cracks in possibly regular meshes due to an appropriate reg-  
33 ularization process, and is highly compatible with TO analysis. Da et al. [32, 33, 34] extended this  
34 work to consider fracture resistance enhancement in composite by considering both interfacial and  
35 bulk fracture. More specifically, the approaches developed in these works considered a full fracture  
36 initiation and propagation within the structure until failure and combined the phase field method  
37 with BESO [35] TO methods. Li et al. [36] extended Xia et al.'s work to the SIMP [37, 38, 3]-  
38 based topology optimization framework and provided a comprehensive comparison of the BESO  
39 and SIMP methods for composite design to brittle fracture resistance. Russ and Waisman [39, 40]  
40 proposed two different topology optimization frameworks for the brittle fracture resistance involv-

41 ing one-phase structure material, combining phase field and TO. In their work, the optimization  
42 problem was defined as minimizing the total volume or volume and fracture energy while ensuring  
43 fracture energy or energy dissipation constraints. Wu et al. [41] developed a level-set method [4, 42]  
44 based topology optimization for the brittle fracture resistance of two-phase composite materials.

45 Taking into account dynamics in the fracture process is of extreme importance for resistance of  
46 structures to impacts. When dynamics are involved, the cracks can interact with wave propagation  
47 and the final crack patterns depend on the energy of the impactor. In addition, dynamics can lead  
48 to more complex crack configurations such as crack branching or initiation of cracks within the  
49 solid. In [43], Miller et al. analyzed the relationship between energy dissipation and crack paths  
50 instabilities in dynamic fracture of brittle materials.

51 The phase field method has been extended to dynamic problems in Borden et al. [44], Hofacker  
52 and Miehe [45], and many extensions and applications have been proposed (see e.g. [46, 47, 48,  
53 49, 50, 51]).

54 In the present work, TO analysis for dynamic fracture resistance of structures is investigated.  
55 As compared to available existing works, the present framework involves the following new con-  
56 tributions: (a) dynamic phase field simulations to fracture is combined with TO for minimizing  
57 the fracture energy of a structure; (b) semi-analytical sensitivities required in the TO analysis in a  
58 dynamic fracture analysis context are provided; (c) comparisons of the obtained fracture resistance  
59 (reduction in the fracture energy) as compared to a static TO designs are discussed.

60 The organization of the paper is as follows. In section 2, the dynamic fracture model involved  
61 in the analysis is reviewed. In section 3, the related time-space discretizations in the context of the  
62 Finite Element Method (FEM) are provided. In section 4, the topology optimization framework  
63 involving the dynamic fracture analysis is developed. The sensitivity analysis related to the fracture  
64 energy of the system in a dynamic context as well as the whole SIMP TO algorithm is provided.  
65 Finally, numerical examples are proposed in section 5 to validate the sensitivity analysis, and to  
66 evaluate the fracture energy reduction of different structures, and more specifically the added value  
67 of the dynamic analysis as compared to a static TO.

## 68 **2. Dynamic phase field fracture model**

69 In this section, the dynamic phase field method for crack propagation in quasi-brittle solids  
70 is briefly reviewed. A structure defined in a domain  $\Omega \subset \mathbb{R}^D$  is considered, with  $D$  the space

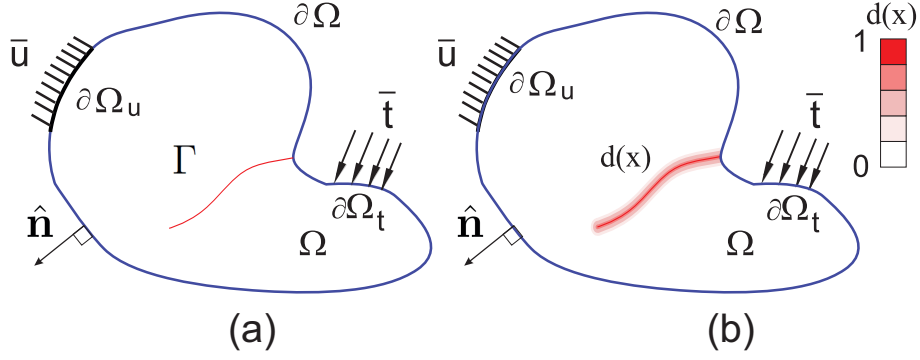


Figure 1: Cracked solid: (a) representation of cracks by surfaces; (b) continuous approximation of cracks by a damage field  $d(\mathbf{x})$ .

71 dimension, with external boundary  $\partial\Omega \subset \mathbb{R}^{D-1}$ . In the context of the phase field method, as shown  
 72 in Fig. 1, the crack surfaces collectively denoted by  $\Gamma$  are described by a continuous damage field  
 73  $d \in [0, 1]$ , which takes 0 value when the material is undamaged and 1 when the material is cracked.  
 74 The portions of  $\partial\Omega$ ,  $\partial\Omega_u$  and  $\partial\Omega_t$  denote the Dirichlet and Neumann boundaries, respectively (see  
 75 Fig. 1). We define  $\mathbf{u}$ ,  $\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt}$  and  $\ddot{\mathbf{u}} = \frac{d^2\mathbf{u}}{dt^2}$  as the displacement, velocity and acceleration vectors,  
 76 respectively.

77 In this context, the elastic strain energy  $E^s$  is defined by

$$E^s(\mathbf{u}, d) = \int_{\Omega} \psi_e(\boldsymbol{\varepsilon}(\mathbf{u}), d) \, d\Omega \quad (1)$$

78 where  $\psi_e$  is a strain density function, whose form will be specified later, and  $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$   
 79 is the linearized second-order strain tensor, with  $\nabla(\cdot)$  the gradient operator. The kinetic energy  
 80 of the solid is defined by:

$$E^k(\dot{\mathbf{u}}) = \int_{\Omega} \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, d\Omega, \quad (2)$$

81 where  $\rho$  is the material density. In the phase field method, a non-local fracture energy is defined  
 82 according to

$$E^f(d) = \int_{\Omega} c_1 \mathcal{G}_c (\omega(d) + \ell^2 \nabla d \cdot \nabla d) \, d\Omega \quad (3)$$

83 where  $c_1$  is a constant,  $\mathcal{G}_c$  is the Griffith-type critical energy release rate,  $\omega(d)$  is a local dam-  
 84 age density function, and  $\ell$  is a length regularization parameter, which defines the width of the  
 85 regularized crack. Finally the work of external forces is defined by

$$W^{ext}(\mathbf{u}) = \int_{\partial\Omega_t} \bar{\mathbf{t}} \cdot \mathbf{u} \, dS + \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\Omega \quad (4)$$

86 where  $\bar{\mathbf{t}}$  denotes prescribed traction over the portion of the boundary  $\partial\Omega_t$  (see Fig. 1), and  $\mathbf{f}$   
 87 denotes body forces. The action-integral over the time interval  $[t_1, t_2]$  is defined by:

$$\mathcal{A} = \int_{t_1}^{t_2} [E^s(\mathbf{u}, d) + E^f(d) - E^k(\dot{\mathbf{u}}) - W^{ext}(\mathbf{u})] \, dt. \quad (5)$$

88 In the dynamic context, the variational principle of nonlocal damage at the core of the phase  
 89 field method implies minimization of the action-integral under the constraint of irreversibility of  
 90 the damage field, i.e.

$$\dot{d} \geq 0. \quad (6)$$

91 where  $\dot{d} = d(d)/dt$  denotes the rate of the damage field.

92 In the following, the different equations of the model in the case of an assumed isotropic quasi-  
 93 brittle solid are specified. We follow Miehe et al. [28] and express the strain density function such  
 94 that damage is induced by traction only as:

$$\psi_e = ((1 - d)^2 + b) \psi_e^+ + \psi_e^-, \quad (7)$$

95 where  $\psi_e^+$  and  $\psi_e^-$  denote the positive and negative components of the strain density function,  
 96 respectively, which can be computed from the strain tensor as

$$\psi_e^\pm = \frac{\lambda}{2} \langle \text{Tr} [\boldsymbol{\varepsilon}] \rangle_\pm^2 + \mu \boldsymbol{\varepsilon}^\pm : \boldsymbol{\varepsilon}^\pm, \quad (8)$$

97 where  $\lambda$  and  $\mu$  are the Lamé coefficients, which can be related to the Young's modulus  $E$  and  
 98 Poisson's ratio  $\nu$  by

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}. \quad (9)$$

99 Note that other decompositions exist (see a comparison and discussion e.g. in [52]). Above,  
 100  $\langle \cdot \rangle_\pm$  can be expressed by  $\langle a \rangle_\pm = \frac{1}{2} (a \pm |a|)$  and  $\text{Tr} [\cdot]$  denotes the trace operator. The positive and  
 101 negative parts  $\boldsymbol{\varepsilon}^\pm$  can be expressed by:

$$\boldsymbol{\varepsilon}^\pm = \sum_{p=1}^D \langle \boldsymbol{\varepsilon}^p \rangle_\pm \mathbf{Q}^p, \quad \mathbf{Q}^p = \mathbf{v}^p \otimes \mathbf{v}^p, \quad (10)$$

102 where  $D$  denotes the space dimension and  $\varepsilon^p$  and  $\mathbf{v}^p$  are the eigenvalues and eigenvectors of  $\boldsymbol{\varepsilon}$ ,  
 103 respectively. The following definitions are used:  $\omega(d) = d^2$  and  $c_1 = \frac{1}{2\ell}$  [27]. Other choices are  
 104 possible, e.g. using  $\omega(d) = d$  and  $c_1 = \frac{3}{8\ell}$  [53]. The first choice induces damage for any loading  
 105 (even though very low at the beginning) while the second choice leads to a linear elastic stage  
 106 before damage. Due to its simplicity, the first choice is adopted here. A more in-depth comparison  
 107 of the different available models and applications in a dynamic context can be found in [54].

108 With these models at hand, stationary variation of (5) leads to the following Euler-Lagrange  
 109 equations:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}, \\ \frac{\mathcal{G}_c}{\ell} (d - \ell^2 \Delta d) = 2(1-d) \psi_e^+, \end{cases} \quad (11)$$

110 where  $\nabla \cdot (\cdot)$  and  $\Delta (\cdot)$  denote the divergence and Laplacian operators, respectively, and where  $\boldsymbol{\sigma}$   
 111 is the Cauchy stress tensor  $\boldsymbol{\sigma} = \frac{\partial \psi_e}{\partial \boldsymbol{\varepsilon}}$ , which is expressed under the above assumptions by:

$$\begin{aligned} \boldsymbol{\sigma} &= ((1-d)^2 + b) \frac{\partial \psi_e^+}{\partial \boldsymbol{\varepsilon}} + \frac{\partial \psi_e^-}{\partial \boldsymbol{\varepsilon}} \\ &= ((1-d)^2 + b) (\lambda \langle \text{Tr} [\boldsymbol{\varepsilon}] \rangle_+ \mathbf{1} + 2\mu \boldsymbol{\varepsilon}^+) + (\lambda \langle \text{Tr} [\boldsymbol{\varepsilon}] \rangle_- \mathbf{1} + 2\mu \boldsymbol{\varepsilon}^-), \end{aligned} \quad (12)$$

112 where  $\mathbf{1}$  is the second-order identity tensor and  $b \ll 1$  a small numerical parameter used to  
 113 maintain stability in the case of fully broken elements. Above,  $\boldsymbol{\varepsilon}^\pm$  and  $\langle \text{Tr} (\boldsymbol{\varepsilon}) \rangle_\pm$  can be related to  
 114  $\boldsymbol{\varepsilon}$  through the following operators:

$$\boldsymbol{\varepsilon}^\pm = \mathbb{P}^\pm : \boldsymbol{\varepsilon} \quad (13)$$

115

$$\langle \text{Tr} [\boldsymbol{\varepsilon}] \rangle_\pm = R^\pm \text{Tr} [\boldsymbol{\varepsilon}] \quad (14)$$

116 in which the components of  $\mathbb{P}^\pm$  are given in closed form as [55]:

$$\mathbb{P}_{ijkl}^\pm := \frac{\partial \boldsymbol{\varepsilon}^\pm}{\partial \boldsymbol{\varepsilon}} = \sum_p^D H(\pm \varepsilon^p) (\mathbf{Q}^p)_{ij} (\mathbf{Q}^p)_{kl} + \frac{1}{2} \sum_p^D \sum_{q \neq p}^D \phi_{pq} \left( (\mathbf{Q}^p)_{ik} (\mathbf{Q}^q)_{jl} + (\mathbf{Q}^p)_{il} (\mathbf{Q}^q)_{jk} \right) \quad (15)$$

117 with

$$\phi_{pq} = \begin{cases} \frac{\langle \varepsilon^p \rangle_\pm - \langle \varepsilon^q \rangle_\pm}{\varepsilon^p - \varepsilon^q}, & \text{if } \varepsilon^p \neq \varepsilon^q \\ H(\pm \varepsilon^p), & \text{if } \varepsilon^p = \varepsilon^q \end{cases} \quad (16)$$

118 where  $H(\cdot)$  denotes the Heaviside step function. The operator  $R^\pm$  is expressed by

$$R^\pm = \frac{1}{2} (\text{sign}(\pm \text{Tr} [\boldsymbol{\varepsilon}]) + 1). \quad (17)$$

120 To achieve non-reversible evolution of cracks, a strain history functional introduced by Miehe  
 121 et al. [28, 45, 56] is adopted to replace  $\psi_e^+$  in Eq. (11), as

$$\mathcal{H}(\mathbf{x}, t) = \max_{s \in (0, t]} \{\psi_e^+(\mathbf{x}, s)\}. \quad (18)$$

122 Then, the equations (11) are substituted by

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}}, \\ \frac{\mathcal{G}_c}{\ell} (d - \ell^2 \Delta d) = 2(1 - d) \mathcal{H}, \end{cases} \quad (19)$$

123 The above equations are completed with boundary conditions as

$$\begin{cases} \mathbf{u} = \bar{\mathbf{u}} & \text{on } \partial\Omega_u, \\ \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \bar{\mathbf{t}} & \text{on } \partial\Omega_t, \\ \nabla d \cdot \hat{\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (20)$$

124 where  $\hat{\mathbf{n}}$  denotes the outward unitary normal vector to  $\partial\Omega$  (see Fig. 1).

125 Multiplying Eqs. (19) by two different test functions,  $\delta \mathbf{u}$  for the displacement problem and  $\delta d$   
 126 for the phase field problem, respectively, integrating the resulting expression over the domain  $\Omega$ ,  
 127 and using the divergence theorem together with boundary conditions yields the associated weak  
 128 forms: find  $d \in H^1(\Omega)$  and find  $\mathbf{u} \in \mathcal{D} = \{\mathbf{v} | \mathbf{v} = \bar{\mathbf{u}} \text{ on } \partial\Omega_u, \mathbf{v} \in H^1(\Omega)\}$  such that:

$$\int_{\Omega} \left( 2\mathcal{H}(\mathbf{u}) + \frac{\mathcal{G}_c}{\ell} \right) d \delta d + \mathcal{G}_c \ell \nabla d \cdot \nabla d \, d\Omega = \int_{\Omega} 2\mathcal{H}(\mathbf{u}) \delta d \, d\Omega \quad (21)$$

$$\int_{\Omega} \{\rho \ddot{\mathbf{u}} \cdot \delta \mathbf{u} + \boldsymbol{\sigma}(\mathbf{u}, d) : \boldsymbol{\varepsilon}(\delta \mathbf{u})\} \, d\Omega - \int_{\partial\Omega_t} \bar{\mathbf{t}} \cdot \delta \mathbf{u} \, dS - \int_{\Omega} \mathbf{f} \cdot \delta \mathbf{u} \, d\Omega = 0 \quad (22)$$

130 for all  $\delta d \in H^1(\Omega)$  and  $\delta \mathbf{u} \in H_0^1(\Omega) = \{\mathbf{v} | \mathbf{v} = 0 \text{ on } \partial\Omega_u, \mathbf{v} \in H^1(\Omega)\}$ ,  $H^1$  is the usual Sobolev  
 131 space of square-integrable derivative functions.

### 132 3. Numerical solving procedure

133 Displacement field  $\mathbf{u}$ , phase field  $d$  and their gradients  $\boldsymbol{\varepsilon}(\mathbf{u})$  and  $\nabla d$  are approximated by  
 134 classical FEM interpolation in the elements of the FEM mesh according to:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{N}_u(\mathbf{x}) \mathbf{u}^e, & \boldsymbol{\varepsilon}(\mathbf{x}) &= \mathbf{B}_u(\mathbf{x}) \mathbf{u}^e, \\ d(\mathbf{x}) &= \mathbf{N}_d(\mathbf{x}) d^e, & \nabla d(\mathbf{x}) &= \mathbf{B}_d(\mathbf{x}) d^e, \end{aligned} \quad (23)$$



135 where  $\mathbf{u}^e$  and  $\mathbf{d}^e$  denote the nodal displacement and phase field in an element  $e$ ,  $\mathbf{N}$  and  $\mathbf{B}$  denote  
 136 the matrices of the shape functions and their derivatives, respectively. The indices ( $u$  and  $d$ ) of  $\mathbf{N}$   
 137 and  $\mathbf{B}$  refer to displacement and phase field variables, respectively.

138 A staggered scheme is adopted. At one time step  $t^n$ , the phase field problem (21) is solved,  
 139 assuming the displacement field  $\mathbf{u}$  given. Then, the mechanical problem (22) is solved assuming the  
 140 phase field  $d$  given. These problems are solved alternatively before solving the problems at the next  
 141 time step. Note that the mechanical problem (22) is nonlinear due to the separated description  
 142 of the strain field in (10). Here, we transform this problem into a linear one by expressing the  
 143 projectors  $\mathbb{P}^\pm$  with respect to the displacements known from the previous time step  $n-1$ , i.e.  
 144  $\mathbb{P}^\pm(\boldsymbol{\varepsilon}^n) \simeq \mathbb{P}^\pm(\boldsymbol{\varepsilon}^{n-1})$ ,  $\mathcal{R}^\pm(\boldsymbol{\varepsilon}^n) \simeq \mathcal{R}^\pm(\boldsymbol{\varepsilon}^{n-1})$ .

145 At time  $t^n$ , the strain history functional described in Eq. (18) can be calculated using

$$\mathcal{H}^n = \begin{cases} (\psi_e^+)^n & \text{if } (\psi_e^+)^n - \mathcal{H}^{n-1} > 0, \\ \mathcal{H}^{n-1} & \text{otherwise,} \end{cases} \quad (24)$$

146 Note that  $\mathcal{H}^n$  is discontinuous, which brings difficulties to the subsequent sensitivity derivations  
 147 presented in section 4. To alleviate this issue, we introduce a continuous version of the history  
 148 function as:

$$\tilde{\mathcal{H}}^n \simeq \mathcal{H}^{n-1} + [(\psi_e^+)^n - \mathcal{H}^{n-1}] g((\psi_e^+)^n - \mathcal{H}^{n-1}) \quad (25)$$

149 where  $g$  is a regularized Heaviside function, defined by

$$g(x) = \frac{1}{2} \left( 1 + \frac{2}{\pi} \arctan \left( \frac{x}{\zeta} \right) \right), \quad (26)$$

150 and  $\zeta$  is regularization parameter. When  $\zeta$  decreases, the approximation is closer to a sharp jump  
 151 (see Fig. 2). In this paper,  $\zeta = 10^{-6}$  is adopted.

152 More specifically, expressing the strain and stress tensors in vector forms in 2D, i.e.  $[\boldsymbol{\varepsilon}] =$   
 153  $[\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]$ ,  $[\boldsymbol{\sigma}] = [\sigma_{11}, \sigma_{22}, \sigma_{12}]$ , the constitutive law (12) can be expressed at time  $t^n$  as:

$$\begin{aligned} [\boldsymbol{\sigma}_n] = & ((1 - d^n)^2 + b) \{ \lambda \mathcal{R}_{n-1}^+ ([\boldsymbol{\varepsilon}^n] \cdot [\mathbf{1}]) [\mathbf{1}] + 2\mu \mathbf{P}_{n-1}^+ [\boldsymbol{\varepsilon}^n] \} \\ & + \lambda \mathcal{R}_{n-1}^- ([\boldsymbol{\varepsilon}^n] \cdot [\mathbf{1}]) [\mathbf{1}] + 2\mu \mathbf{P}_{n-1}^- [\boldsymbol{\varepsilon}^n], \end{aligned} \quad (27)$$

154 where  $\mathcal{R}_{n-1}^\pm = \mathcal{R}^\pm(\boldsymbol{\varepsilon}^{n-1})$  and  $\mathbf{P}_{n-1}^\pm = \mathbf{P}^\pm(\boldsymbol{\varepsilon}^{n-1})$ , and  $\mathbf{P}^\pm$  are the matrix forms associated with  
 155 the fourth-order tensors  $\mathbb{P}^\pm$ .

157 Introducing (23) and (27) in (21)-(22) we obtain a linear system of equations in the form

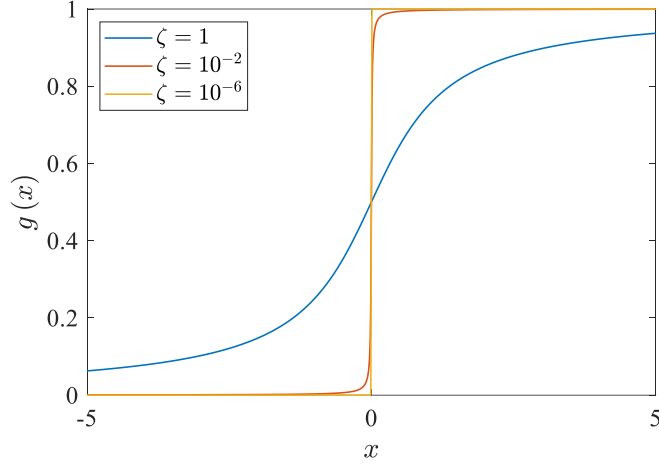


Figure 2: Regularized Heaviside function.

$$\mathbf{K}_d^n \mathbf{d}^n = \mathbf{F}_d^n, \quad (\text{phase field problem}), \quad (28)$$

$$\mathbf{M} \ddot{\mathbf{u}}^n + \mathbf{K}_u^n \mathbf{u}^n = \mathbf{F}_u^n, \quad (\text{displacement problem}) \quad (29)$$

158 with

$$\mathbf{M} = \int_{\Omega} \rho \mathbf{N}_u^T \mathbf{N}_u d\Omega \quad \text{and} \quad \mathbf{F}_u^n = \int_{\partial\Omega_{\bar{t}}} \mathbf{N}_u \bar{\mathbf{t}}^n dS. \quad (30)$$

$$\begin{aligned} \mathbf{K}_u^n = & \int_{\Omega} \mathbf{B}_u^T \left\{ ((1 - d^n)^2 + b) \left( \lambda \mathcal{R}_{n-1}^+ [\mathbf{1}]^T [\mathbf{1}] + 2\mu \mathbf{P}_{n-1}^+ \right) \right\} \mathbf{B}_u d\Omega, \\ & + \int_{\Omega} \mathbf{B}_u^T \left\{ \lambda \mathcal{R}_n^- [\mathbf{1}]^T [\mathbf{1}] + 2\mu \mathbf{P}_n^- \right\} \mathbf{B}_u d\Omega, \end{aligned} \quad (31)$$

$$\mathbf{K}_d^n = \int_{\Omega} \left\{ \left( \frac{\mathcal{G}_c}{\ell} + 2\mathcal{H}^n \right) \mathbf{N}_d^T \mathbf{N}_d + \mathcal{G}_c \ell \mathbf{B}_d^T \mathbf{B}_d \right\} d\Omega \quad (32)$$

159 and

$$\mathbf{F}_d^n = \int_{\Omega} 2\mathbf{N}_d^T \mathcal{H}^n d\Omega. \quad (33)$$

160 Then, a time-stepping  $\mathcal{I} = [t^0, t^1, t^2, \dots, t^M]$  is introduced, where  $t^n - t^{n-1} = \Delta t$  is a time step,  
 161 assumed to be constant. An unconditionally stable implicit Newmark scheme is used to solve (29)  
 162 according to:

$$\dot{\mathbf{u}}^n = \dot{\mathbf{u}}^{n-1} + \frac{\Delta t}{2} (\ddot{\mathbf{u}}^{n-1} + \ddot{\mathbf{u}}^n), \quad (34)$$

163 with

$$\mathbf{u}^n = \mathbf{u}^{n-1} + \Delta t \dot{\mathbf{u}}^{n-1} + \left( \frac{\Delta t^2}{4} \right) (\ddot{\mathbf{u}}^{n-1} + \ddot{\mathbf{u}}^n), \quad (35)$$

$$\ddot{\mathbf{u}}^n = \left( \frac{4}{\Delta t^2} \right) (\mathbf{u}^n - \hat{\mathbf{u}}^n) \quad (36)$$

$$\hat{\mathbf{u}}^n = \mathbf{u}^{n-1} + \Delta t \dot{\mathbf{u}}^{n-1} + \left( \frac{\Delta t^2}{4} \right) \ddot{\mathbf{u}}^{n-1}. \quad (37)$$

164 Introducing (36) into (29), we finally obtain at one iteration  $k$  and at one time step  $t^n$  the  
165 linear problem to be solved:

$$\tilde{\mathbf{K}}_u \mathbf{u}^n = \tilde{\mathbf{F}}_u^n, \quad (38)$$

166 with

$$\tilde{\mathbf{K}}_u = \left( \frac{4}{\Delta t^2} \right) \mathbf{M} + \mathbf{K}_u^n \quad (39)$$

167

$$\tilde{\mathbf{F}}_u^n = \mathbf{F}_u^n + \mathbf{M} \left( \frac{4}{\Delta t^2} \right) \left[ \mathbf{u}^{n-1} + \Delta t \dot{\mathbf{u}}^{n-1} + \left( \frac{\Delta t}{2} \right)^2 \ddot{\mathbf{u}}^{n-1} \right]. \quad (40)$$

168 Note that in the present work iterations are used within the staggered scheme, i.e the mechanical  
169 and phase field problems are solved alternatively during one time step  $t^n$  until a convergence  
170 criterion is reached. In this work, the convergence criterion is reached when the maximum value  
171 of the nodal phase field variation between two iterations is lower than a threshold. This algorithm  
172 allows larger time steps in the present implicit Newmark's scheme. In addition, such staggered  
173 scheme is a key ingredient to simplify the sensitivity analysis developed in section 4.3.

174 The general algorithm is summarized in Algorithm 1, where quantities at one time step  $t^n$  and  
175 at one iteration  $k$  are denoted by  $(\cdot)_k^n$ .

## 176 4. Topology optimization formulations

### 177 4.1. Material interpolation scheme

178 In the present paper, the SIMP topology optimization method (Solid Isotropic Material with  
179 Penalization (SIMP) method [3, 37]) is adopted. This method belongs to the so-called density-  
180 based topology optimization (TO) methods (see a review and classification of TO in [57]). In

---

**Algorithm 1:** Staggered dynamic Phase Field algorithm solving procedure.

---

Initialize:  $\mathbf{u}^0, \dot{\mathbf{u}}^0, \ddot{\mathbf{u}}^0, \mathbf{d}^0, \mathcal{H}^0 = \mathbf{0}$ ;

**for**  $n = 1 : M$  (*Loop over all time steps*) **do**

**Input:**  $\mathbf{u}^{n-1}$

    Initialize  $\mathbf{u}_0^n = \mathbf{u}^{n-1}$

$k = k + 1$

**while**  $Err > tol$  (*Convergence loop*) **do**

$k = k + 1$ ;

**Input:**  $\mathbf{u}_{k-1}^n$

        Solve  $\mathbf{K}_d(\mathbf{u}_{k-1}^n)\mathbf{d}_k^n = \mathbf{F}_d(\mathbf{u}_{k-1}^n)$

**Output:**  $\mathbf{d}_k^n$

**Input:**  $\mathbf{d}_k^n, \mathbf{u}_{k-1}^n$

        Solve  $\mathbf{K}_u(\mathbf{u}_{k-1}^n, \mathbf{d}_k^n)\mathbf{u}_k^n = \mathbf{F}_u(\mathbf{u}_{k-1}^n)$

**Output:**  $\mathbf{u}_k^n$

        Compute  $Err = \max_j |[\mathbf{d}_k^n]_j - [\mathbf{d}_{k-1}^n]_j|$

$\mathbf{u}_{k-1}^n = \mathbf{u}_k^n$

**end**

$\mathbf{u}^n = \mathbf{u}_k^n$

**end**

---

181 this framework, the geometry of the structural domain is defined by a pseudo-density parameter  
 182  $\varphi \in [0, 1]$ . Then, a continuous description of the material properties is defined according to:

$$\begin{aligned}
 E(\varphi) &= [E_{\min} + (1 - E_{\min}) \varphi^{p_E}] E_0, \\
 \rho(\varphi) &= [\rho_{\min} + (1 - \rho_{\min}) \varphi^{p_\rho}] \rho_0, \\
 \mathcal{G}_c(\varphi) &= [\mathcal{G}_{c,\min} + (1 - \mathcal{G}_{c,\min}) \varphi^{p_{\mathcal{G}}}] \mathcal{G}_{c,0},
 \end{aligned} \tag{41}$$

183 where  $E_0$ ,  $\rho_0$  and  $\mathcal{G}_{c,0}$  denote the material properties of the solid for  $\varphi = 1$ , and  $E_{\min}$ ,  $\rho_{\min}$  and  
 184  $\mathcal{G}_{c,\min}$  are artificial lower values to avoid numerical singularity in zero-pseudo density zones. Above,  
 185  $p_E$ ,  $p_\rho$  and  $p_{\mathcal{G}}$  are penalty parameters. These parameters are chosen so as to avoid intermediate  
 186 values of  $\varphi$ . Here, we choose  $p_E = 3$ ,  $p_\rho = 1$  and  $p_{\mathcal{G}} = 1$ . The values of  $E_{\min}$ ,  $\rho_{\min}$  and  $\mathcal{G}_{c,\min}$   
 187 are defined as  $E_{\min} = 10^{-6}$ ,  $\rho_{\min} = 10^{-6}$  and  $\mathcal{G}_{c,\min} = 10^{-2}$ , respectively. Note that alternative  
 188 approaches, e.g. the BESO method [35] could be used (see [36] for a comparison between SIMP  
 189 and BESO in the context of fracture resistance maximization).

#### 190 4.2. Optimization problem

191 In this section, we define the topology optimization problem related to minimizing the dynamic  
 192 fracture of a structure. Following previous works on fracture resistance maximization using topol-  
 193 ogy optimization [25, 36], the phase field method described in section 3 is used to describe the full  
 194 fracture process of the structure, from initiation until full failure, while here the dynamic effects  
 195 are taken into account. The problem can be formulated as follows:

$$\begin{aligned}
 \min_{\varphi \in [0,1]} & : G_f(\varphi), \\
 \text{s. t.} & : \mathbf{K}_d^n \mathbf{d}^n = \mathbf{F}_d^n, \quad n = 1, 2, \dots, M \\
 & \mathbf{M} \ddot{\mathbf{u}}^n + \mathbf{K}_u^n \mathbf{u}^n = \mathbf{F}_u^n, \quad n = 1, 2, \dots, M \\
 & C - C_0 \leq 0, \\
 & V - \chi V_0 \leq 0,
 \end{aligned} \tag{42}$$

196 where  $G_f(\varphi)$  denotes the fracture energy, and  $C = \mathbf{F}^s \cdot \mathbf{u}^s$  denotes a structural static compliance.

197 Here, the compliance constraint is only used to ensure connectivity of the material within the  
 198 structure. Without this constraint, unrealistic topologies with disconnected parts could be obtained  
 199 in this dynamic context, as cracks may occur in the middle of the structure. The compliance  $C$  is  
 200 evaluated by a separated static test with an external force  $F^s = -100$  N. Note that such compliance

201 constraint has been used by several other authors in a dynamic topology optimization context,  
 202 even though in a linear vibration regime (see e.g. [58, 59]). Above,  $\mathbf{u}^s$  is the static displacement  
 203 response of the undamaged structure under a test static external load  $\mathbf{F}^s$  and  $C_0$  is the prescribed  
 204 compliance upper bound;  $V$  is the target structural volume (area in 2D), which can be computed  
 205 as  $V = \sum_{e=1}^{N_e} V_e \varphi_e$ , and  $V_0 = \sum_{e=1}^{N_e} V_e$  is the total volume of the design domain, and  $\chi$  denotes the  
 206 target volume fraction constraint. We define the fracture energy over the whole loading history as:

$$G_f = \int_0^T \int_{\Omega} \left( \frac{1}{\ell} d \cdot \dot{d} + \ell \nabla d \cdot \nabla \dot{d} \right) \mathcal{G}_c \, d\Omega \, dt. \quad (43)$$

207 Using the trapezoidal rule,  $G_f$  can be numerically approximated as

$$G_f = \sum_{n=1}^M \frac{1}{2} \left[ (\mathbf{d}^n)^T \mathbf{K}_g \dot{\mathbf{d}}^n + (\mathbf{d}^{n-1})^T \mathbf{K}_g \dot{\mathbf{d}}^{n-1} \right] \Delta t, \quad (44)$$

208 where  $\mathbf{K}_g$  is defined by

$$\mathbf{K}_g = \int_{\Omega} \mathcal{G}_c \left( \frac{1}{\ell} (\mathbf{N}_d)^T \mathbf{N}_d + \ell (\mathbf{B}_d)^T \mathbf{B}_d \right) \, d\Omega. \quad (45)$$

209 Above,  $\mathbf{K}_g$  is a matrix which depends neither on damage nor on displacement fields.

### 210 4.3. Sensitivity of fracture energy

211 In this section, the sensitivity of fracture energy is derived in a dynamic context. The deriva-  
 212 tions require the use of the adjoint method [60, 61, 62]. For the widely known derivations related  
 213 to the compliance and volume fraction, one may refer to [63, 64] for details.

214 The sensitivity of  $G_f$  with respect to a change in the pseudo-density is given by

$$\frac{\partial G_f}{\partial \varphi_e} = \sum_{n=1}^M \frac{1}{2} \frac{\partial}{\partial \varphi_e} \left[ (\mathbf{d}^n)^T \mathbf{K}_g \dot{\mathbf{d}}^n + (\mathbf{d}^{n-1})^T \mathbf{K}_g \dot{\mathbf{d}}^{n-1} \right] \Delta t, \quad (46)$$

215 and involves evaluating  $\frac{\partial \mathbf{d}^n}{\partial \varphi_e}$  and  $\frac{\partial \dot{\mathbf{d}}^n}{\partial \varphi_e}$ . Using the chain rule, we have

$$\frac{\partial \dot{\mathbf{d}}^n}{\partial \varphi_e} = \frac{\partial \dot{\mathbf{d}}^n}{\partial \mathbf{d}^n} \frac{\partial \mathbf{d}^n}{\partial \varphi_e} \quad (47)$$

216 where  $\frac{\partial \dot{\mathbf{d}}^n}{\partial \mathbf{d}^n}$  can be obtained by the Newmark scheme (34)-(36)(37) as the simple expression:

$$\frac{\partial \dot{\mathbf{d}}^n}{\partial \mathbf{d}^n} = \frac{2}{\Delta t}. \quad (48)$$

217 To express  $\frac{\partial \mathbf{d}^n}{\partial \varphi_e}$  the adjoint method [62] is employed. Introducing two vectors of Lagrange  
 218 multipliers (adjoint vectors)  $\boldsymbol{\lambda}^n$  and  $\boldsymbol{\lambda}^{n-1}$ , and assuming that the problems

219

$$\mathbf{R}_d^n = \mathbf{K}_d^n \mathbf{d}^n - \mathbf{F}_d^n = \mathbf{0} \quad (49)$$

$$\mathbf{R}_d^{n-1} = \mathbf{K}_d^n \mathbf{d}^{n-1} - \mathbf{F}_d^{n-1} = \mathbf{0} \quad (50)$$

220 have been solved, then the terms  $(\boldsymbol{\lambda}^n)^T \mathbf{R}_d^n$  and  $(\boldsymbol{\lambda}^{n-1})^T \mathbf{R}_d^{n-1}$  can be added to the objective  
221 function without change as:

$$G_f = \sum_{n=1}^{N_s} \left\{ \frac{1}{2} \left[ (\mathbf{d}^n)^T \mathbf{K}_g \dot{\mathbf{d}}^n + (\mathbf{d}^{n-1})^T \mathbf{K}_g \dot{\mathbf{d}}^{n-1} \right] \Delta t + (\boldsymbol{\lambda}^n)^T \mathbf{R}_d^n + (\boldsymbol{\lambda}^{n-1})^T \mathbf{R}_d^{n-1} \right\}. \quad (51)$$

222 In addition, using (8),  $(\psi_e^+)^n$  can be expressed using the discrete (vector) forms of strain tensor  
223 as:

$$(\psi_e^+)^n = \frac{1}{2} \boldsymbol{\varepsilon}^n \left( \lambda \mathcal{R}_{n-1}^+ [\mathbf{1}]^T [\mathbf{1}] + 2\mu \mathbf{P}_{n-1}^+ \right) \boldsymbol{\varepsilon}^n. \quad (52)$$

224 In (52), it is worth noting that  $\varepsilon$  obviously depends on  $\varphi_e$ . However, for the sake of simplicity,  
225 we assume that the term involving  $\frac{\partial \boldsymbol{\varepsilon}}{\partial \varphi_e}$  has small influence as compared to the other terms and  
226 neglect it. Then, the following approximation is made:

$$\frac{\partial (\psi_e^+)^n}{\partial \varphi_e} \simeq \frac{1}{2} \boldsymbol{\varepsilon}^n \left( \frac{\partial \lambda}{\partial \varphi_e} \mathcal{R}_{n-1}^+ [\mathbf{1}]^T [\mathbf{1}] + 2 \frac{\partial \mu}{\partial \varphi_e} \mathbf{P}_{n-1}^+ \right) \boldsymbol{\varepsilon}^n. \quad (53)$$

227

228 Taking the derivation of  $G_f$  with respect to the pseudo-density, using (48) and combining similar  
229 terms, the following expression is obtained, after some calculations:

$$\begin{aligned} \frac{\partial \hat{G}_f}{\partial \varphi_e} = & \sum_{n=1}^{N_s} \left\{ \frac{1}{2} \left[ (\mathbf{d}^n)^T \frac{\partial \mathbf{K}_g}{\partial \varphi_e} \dot{\mathbf{d}}^n + (\mathbf{d}^{n-1})^T \frac{\partial \mathbf{K}_g}{\partial \varphi_e} \dot{\mathbf{d}}^{n-1} \right] \Delta t \right. \\ & + (\boldsymbol{\lambda}^n)^T \left( \frac{\partial \mathbf{K}_d^n}{\partial \varphi_e} \mathbf{d}^n - \frac{\partial \mathbf{F}_d^n}{\partial \varphi_e} \right) + (\boldsymbol{\lambda}^{n-1})^T \left( \frac{\partial \mathbf{K}_d^{n-1}}{\partial \varphi_e} \mathbf{d}^{n-1} - \frac{\partial \mathbf{F}_d^{n-1}}{\partial \varphi_e} \right) \\ & + \left[ \frac{1}{2} \left( (\dot{\mathbf{d}}^n)^T \mathbf{K}_g + (\mathbf{d}^n)^T \mathbf{K}_g \frac{\partial \dot{\mathbf{d}}^n}{\partial \varphi_e} \right) \Delta t + (\boldsymbol{\lambda}^n)^T \mathbf{K}_d^n \right] \frac{\partial \mathbf{d}^n}{\partial \varphi_e} \\ & \left. + \left[ \frac{1}{2} \left( (\dot{\mathbf{d}}^{n-1})^T \mathbf{K}_g + (\mathbf{d}^{n-1})^T \mathbf{K}_g \frac{\partial \dot{\mathbf{d}}^{n-1}}{\partial \varphi_e} \right) \Delta t + (\boldsymbol{\lambda}^{n-1})^T \mathbf{K}_d^{n-1} \right] \frac{\partial \mathbf{d}^{n-1}}{\partial \varphi_e} \right\}. \end{aligned} \quad (54)$$

230 The terms  $\frac{\partial \mathbf{d}^n}{\partial \varphi_e}$  and  $\frac{\partial \mathbf{d}^{n-1}}{\partial \varphi_e}$  are difficult to evaluate in practice. However, as  $\mathbf{R}_d^n = \mathbf{0}$  and  
231  $\mathbf{R}_d^{n-1} = \mathbf{0}$ , the vectors  $\boldsymbol{\lambda}^n$  and  $\boldsymbol{\lambda}^{n-1}$  can be chosen arbitrarily. They are then chosen to eliminate  
232 the unknown terms  $\frac{\partial \mathbf{d}^n}{\partial \varphi_e}$  and  $\frac{\partial \mathbf{d}^{n-1}}{\partial \varphi_e}$  such that:

$$\left[ \frac{1}{2} \left( (\dot{\mathbf{d}}^n)^T \mathbf{K}_g + (\mathbf{d}^n)^T \mathbf{K}_g \frac{\partial \dot{\mathbf{d}}^n}{\partial \mathbf{d}^n} \right) \Delta t + (\boldsymbol{\lambda}^n)^T \mathbf{K}_d^n \right] \frac{\partial \mathbf{d}^n}{\partial \varphi_e} = 0 \quad (55)$$

233 and

$$\left[ \frac{1}{2} \left( (\dot{\mathbf{d}}^{n-1})^T \mathbf{K}_g + (\mathbf{d}^{n-1})^T \mathbf{K}_g \frac{\partial \dot{\mathbf{d}}^{n-1}}{\partial \mathbf{d}^{n-1}} \right) \Delta t + (\boldsymbol{\lambda}^{n-1})^T \mathbf{K}_d^{n-1} \right] \frac{\partial \mathbf{d}^{n-1}}{\partial \varphi_e} = 0. \quad (56)$$

234 Eqs. (55) and (56) are equal to zero if the expressions under brackets on the left-hand are equal  
235 to zero, corresponding to the following systems of equations:

$$2\mathbf{K}_d^n \boldsymbol{\lambda}^n = - \left( \mathbf{K}_g \dot{\mathbf{d}}^n + \mathbf{K}_g \frac{\partial \dot{\mathbf{d}}^n}{\partial \mathbf{d}^n} \mathbf{d}^n \right) \Delta t, \quad (57)$$

236 and

$$2\mathbf{K}_d^{n-1} \boldsymbol{\lambda}^{n-1} = - \left( \mathbf{K}_g \dot{\mathbf{d}}^{n-1} + \mathbf{K}_g \frac{\partial \dot{\mathbf{d}}^{n-1}}{\partial \mathbf{d}^{n-1}} \mathbf{d}^{n-1} \right) \Delta t. \quad (58)$$

237 Solving Eqs. (57) and (58), the Lagrange multipliers  $\boldsymbol{\lambda}^n$  and  $\boldsymbol{\lambda}^{n-1}$  are then available. Above,  
238 the expressions of  $\dot{\mathbf{d}}^n$  can be computed according to (34). The sensitivity of the fracture energy is  
239 then finally obtained as:

$$\boxed{\frac{\partial G_f}{\partial \varphi_e} = \sum_{n=1}^{N_s} \left\{ \frac{1}{2} \left[ (\mathbf{d}^n)^T \frac{\partial \mathbf{K}_g}{\partial \varphi_e} \dot{\mathbf{d}}^n + (\mathbf{d}^{n-1})^T \frac{\partial \mathbf{K}_g}{\partial \varphi_e} \dot{\mathbf{d}}^{n-1} \right] \Delta t + (\boldsymbol{\lambda}^n)^T \left( \frac{\partial \mathbf{K}_d^n}{\partial \varphi_e} \mathbf{d}^n - \frac{\partial \mathbf{F}_d^n}{\partial \varphi_e} \right) + (\boldsymbol{\lambda}^{n-1})^T \left( \frac{\partial \mathbf{K}_d^{n-1}}{\partial \varphi_e} \mathbf{d}^{n-1} - \frac{\partial \mathbf{F}_d^{n-1}}{\partial \varphi_e} \right) \right\}} \quad (59)$$

240 in which  $\frac{\partial \mathbf{K}_g}{\partial \varphi_e}$  can be derived from Eq. (45), as:

$$\frac{\partial \mathbf{K}_g}{\partial \varphi_e} = \int_{\Omega} \frac{\partial \mathcal{G}_c}{\partial \varphi_e} \left( \frac{1}{\ell} (\mathbf{N}_d)^T \mathbf{N}_d + \ell (\mathbf{B}_d)^T \mathbf{B}_d \right) d\Omega. \quad (60)$$

241 The terms  $\frac{\partial \mathbf{K}_d^n}{\partial \varphi_e}$  and  $\frac{\partial \mathbf{F}_d^n}{\partial \varphi_e}$  are given by

$$\begin{aligned} \frac{\partial \mathbf{K}_d^n}{\partial \varphi_e} &= \int_{\Omega} \left\{ \left( 2 \frac{\partial \mathcal{H}_n}{\partial \varphi_e} + \frac{\partial \mathcal{G}_c}{\ell \partial \varphi_e} \right) (\mathbf{N}_d)^T \mathbf{N}_d + \frac{\partial \mathcal{G}_c}{\partial \varphi_e} \ell (\mathbf{B}_d)^T \mathbf{B}_d \right\} d\Omega, \\ \frac{\partial \mathbf{F}_d^n}{\partial \varphi_e} &= \int_{\Omega} 2 \frac{\partial \mathcal{H}_n}{\partial \varphi_e} \mathbf{N}_d d\Omega, \end{aligned} \quad (61)$$

242 where  $\frac{\partial \mathcal{G}_c}{\partial \varphi_e}$  can be obtained from the material interpolation scheme (41), and  $\frac{\partial \mathcal{H}_n}{\partial \varphi_e}$  can be derived  
243 from Eq. (25):

$$\frac{\partial \mathcal{H}^n}{\partial \varphi_e} = \frac{\partial \psi_e^+}{\partial \varphi_e} g(\psi_e^+ - \mathcal{H}^{n-1}) + [\psi_e^+ - \mathcal{H}^{n-1}] \frac{\partial g(\psi_e^+ - \mathcal{H}^{n-1})}{\partial \varphi_e} \quad (62)$$



244 with

$$\frac{\partial g((\psi_e^+)^n - \mathcal{H}^{n-1})}{\partial \varphi_e} = \frac{\zeta \frac{\partial (\psi_e^+)^n}{\partial \varphi_e}}{\pi \left( \zeta^2 + ((\psi_e^+)^n - \mathcal{H}^{n-1})^2 \right)}, \quad (63)$$

245 and the term  $\frac{\partial (\psi_e^+)^n}{\partial \varphi_e}$  have been approximated by Eq. (53). Note that above adjoint vectors  $\boldsymbol{\lambda}^n$  and  
 246  $\boldsymbol{\lambda}^{n-1}$  are here path-independent, in contrast to other formulations, see e.g. [39, 40]. This strong  
 247 assumption has the advantage to gratefully simplify the formulation and the implementation. The  
 248 influence of such simplification on the accuracy of the sensitivities will be tested in the numerical  
 249 examples.

#### 250 4.4. Optimization techniques

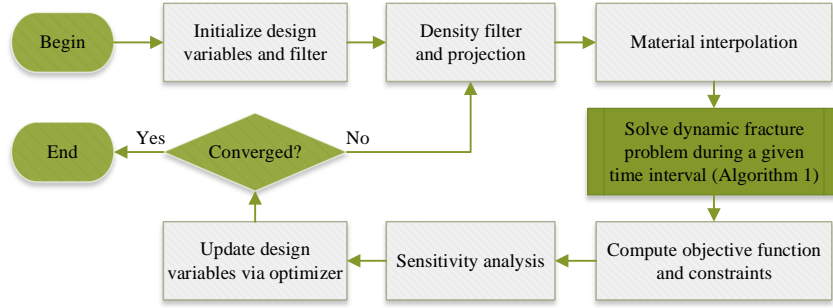


Figure 3: Optimization flow chart.

251 To improve stability, mesh independence and to eliminate so called *checkerboard* issues [65],  
 252 filtering techniques are often used in topology optimization. Following [66, 64], a filtered density  
 253 variable  $\theta_e$  is introduced as

$$\theta_e = \frac{\sum_{i \in \mathcal{N}_e} \varpi_e V_e \vartheta_e}{\sum_{i \in \mathcal{N}_e} \varpi_e V_e} \quad (64)$$

254 where  $\vartheta$  denotes the design variable and  $\mathcal{N}_e$  denotes the set of elements whose center-to-center  
 255 distance  $r_{ei}$  to the  $e$ -th element is lower than the filter radius  $r_{\min}$ . The corresponding weighting  
 256 factor  $\varpi_{ei}$  is defined by  $\varpi_{ei} = \max(0, r_{\min} - r_{ei})$ .

257 The projection technique proposed by Guest et al. [67] is then adopted to minimize transition  
 258 regions with pseudo-density values  $\varphi$  between zero and one, as

$$\varphi_e = 1 - e^{-\eta \theta_e} + \theta_e e^{-\eta} \quad (65)$$

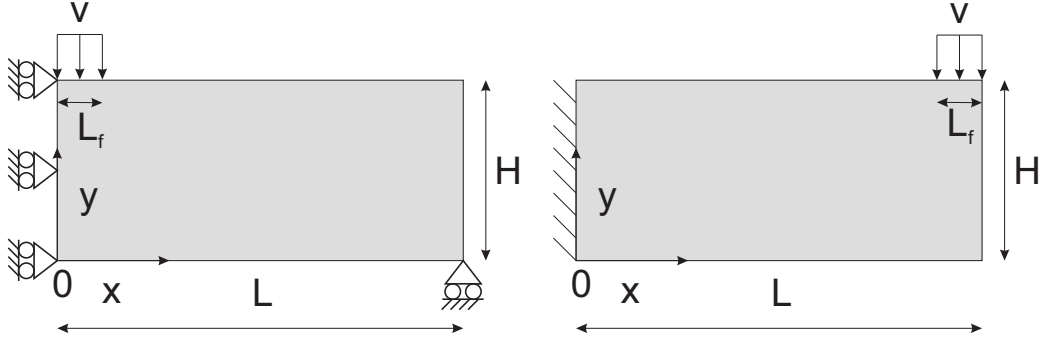


Figure 4: Geometry and boundary conditions of: (a) Half MBB-beam; (b) Cantilever beam.

259 where  $\varphi_e$  is the elemental pseudo-density, and  $\eta$  is a parameter defined by 1 in the first iteration  
 260 and is doubled after every specified time steps until it reaches a chosen maximum value, taken  
 261 here as 128 by numerical tests.

262 Using this procedure, the sensitivities of the objective functions and optimization constraints  
 263 with respect to the design variable can be further derived by means of the chain rule as

$$\frac{\partial f(\varphi_e)}{\partial \vartheta_e} = \frac{\partial f}{\partial \varphi_e} \frac{\partial \varphi_e}{\partial \theta_e} \frac{\partial \theta_e}{\partial \vartheta_e}. \quad (66)$$

264 The method of moving asymptotes (MMA) proposed by Svanberg [68] is adopted for seeking  
 265 the optimal distribution of the design variables  $\vartheta$ . Following [69], the  $\eta$ -based modification on the  
 266 asymptotes are adopted for removing spurious oscillations after doubling the projection parameter  
 267  $\eta$ . The convergence criterion of the topology optimization is determined by the maximal change  
 268 on the design variable, which should be less than  $10^{-3}$ . Fig. (3) summarizes the flow chart of the  
 269 proposed topology optimization.

## 270 5. Numerical examples

271 In this section, two typical 2D structures are considered. These have been widely studied in the  
 272 topology optimization community. The first one is a 3-point bending beam, also called MBB-beam  
 273 in the literature [70]. For the sake of computational costs, only the right half of this axisymmetric  
 274 beam is considered as shown in Fig. 4(a). The left end is simply supported in the  $x$ -direction  
 275 and the lower right-end corner is simply supported in the  $y$ -direction. The second structure is a  
 276 cantilever beam, which is shown in Fig. 4(b). The length and width of these two structures are the  
 277 same,  $L = 150$  mm and  $H = 60$  mm. A velocity is prescribed on a surface of length  $L_f = 4$  mm.  
 278 Fig. 5 depicts the loading velocity profile, which increases from 0 to  $v_0$  by a time  $t_0$ , and then

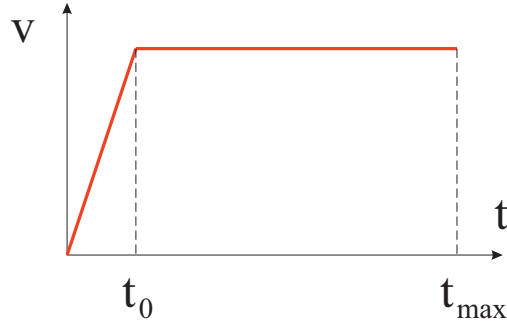


Figure 5: Prescribed velocity.

Table 1: Material parameters.

Material properties	Symbol	Value	Unit
Young's modulus	$E_0$	$1.9 \times 10^5$	MPa
Poisson's ratio	$\nu$	0.3	-
Density	$\rho_0$	$8 \times 10^{-9}$	ton/mm <sup>3</sup>
Critical energy release rate	$\mathcal{G}_{c,0}$	22.17	N/mm

279 remains constant until the maximum time  $t_{\max}$  is reached. The material properties adopted here  
 280 are taken from the Kalthoff-Winkler experiment [71], and are summarized in Table 1. The same  
 281 geometry, loading curve and parameters will be kept in all following examples. In the example of  
 282 section 5.1, the structure is discretized into a coarse  $75 \times 30$  four-node quadrilateral elements mesh  
 283 for the sake of computational costs. In the examples of sections 5.2 and 5.3, a finer mesh with  
 284  $150 \times 60$  four-node quadrilateral finite elements is adopted.

285 To evaluate the added value of the present framework, two solutions are defined:

- 286 1. A so-called "S-design" solution. This solution is obtained by static topology optimization  
 287 with minimization of compliance under volume constraint with a static force chosen as  $\mathbf{F}^s =$   
 288  $-100\text{N}$ . Then, the design is remained unchanged during the dynamic fracture simulation.

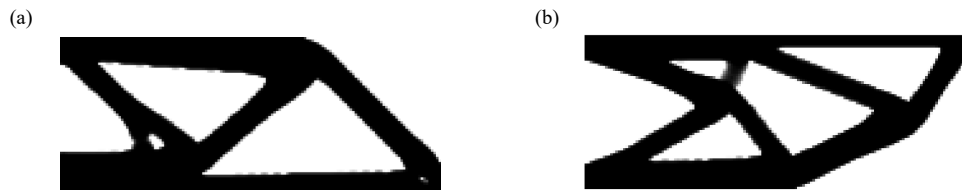


Figure 6: S-designs obtained for :(a) the half MBB-beam; (b) the cantilever beam.

Table 2: Numerical parameters for validation of sensitivity analysis.

$\varphi$	$\Delta\varphi_e$	$\ell$	$v_0$	$\Delta t$	$t_0$	$t_{\max}$	$tol$
0.5	$10^{-6}$	4 mm	40m/s	$1 \mu s$	$2 \mu s$	$80 \mu s$	$10^{-5}$

The obtained design for the two problems studied in the next examples, namely the half MBB-beam and the cantilever beam are depicted in Fig. 6.

- A so-called "DF-design" solution. In that case, the problem 42 is solved to define the topology: at each iteration of the algorithm, a static problem is firstly solved to prescribe the compliance constraint, then a full dynamic fracture simulation is performed to evaluate the fracture energy, and compute the sensitivities to update the topology.

### 5.1. Validation of sensitivity analysis

First the sensitivity analysis developed in section 4.3 is validated. Both half MBB-beam and cantilever beam are considered.

The central finite difference method is employed to provide a reference solution to be compared with our semi-analytical sensitivities expressions, according to:

$$\frac{df(\varphi)}{d\varphi_e} \approx \frac{f(\varphi_1, \dots, \varphi_e + \Delta\varphi_e, \dots, \varphi_{N_e}) - f(\varphi_1, \dots, \varphi_e - \Delta\varphi_e, \dots, \varphi_{N_e})}{2\Delta\varphi_e} \quad (67)$$

where  $\Delta\varphi_e$  is a pseudo-density perturbation parameter. The value of the numerical parameters are listed on Table 2.

Figs. 7(a) and (b) depict the fracture plot of the half MBB-beam and cantilever beam at time  $t_{\max} = 80 \mu s$ , in which only  $d > 0.6$  is depicted for the sake of clarity. Figs. 7(c) and (d) show the normalized error map of the sensitivity values, which is defined by:

$$error = \frac{|\xi^{dif} - \xi^{ana}|}{|\overline{\xi^{ana}}|} \quad (68)$$

where  $\xi$  denotes the vector of element sensitivity values. The superscripts *dif* and *ana* indicate the finite difference method and semi-analytical method, respectively, and  $\overline{\xi^{ana}}$  denotes the maximum element sensitivity obtained by the semi-analytical method. Figs. 7(e) and (f) show a comparison between elemental sensitivities associated with the fracture energy of these two structures. A good agreement between our analytical expressions of sensitivities and the reference finite difference solution is noticed.

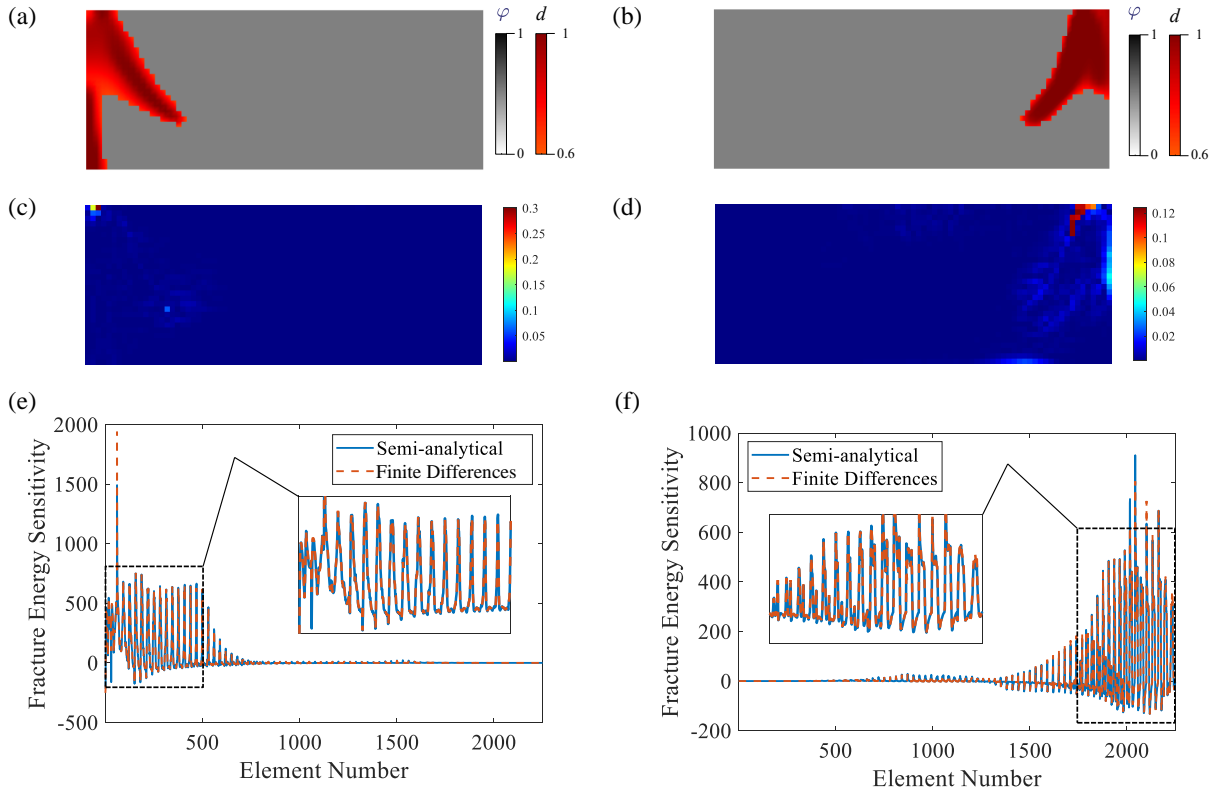


Figure 7: Validation of fracture energy sensitivity analysis: fracture pattern at  $t_{\max} = 80 \mu\text{s}$  in (a) the half MBB-beam; (b) the cantilever beam; error of normalized sensitivity values (c) the half MBB-beam; (d) the cantilever beam; sensitivity values in elements at  $t_{\max} = 80$  in (e) the half MBB-beam and (f) the cantilever beam.

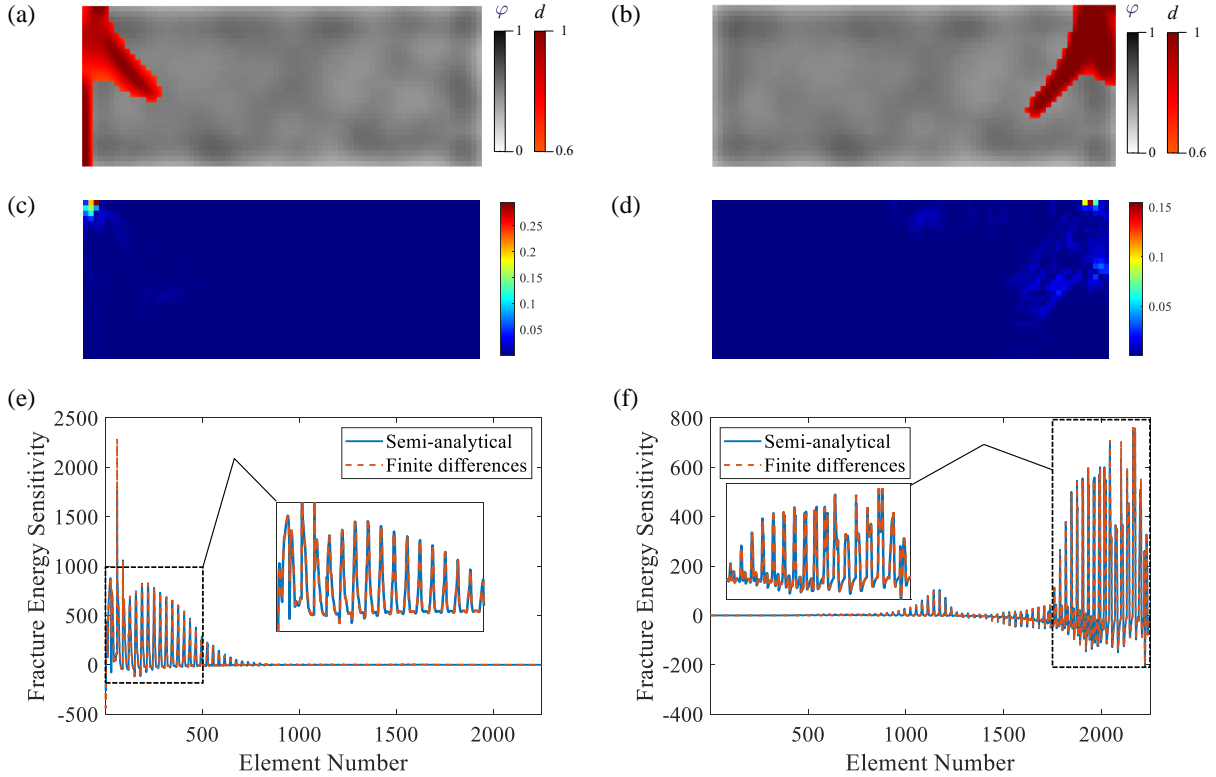


Figure 8: Validation of fracture energy sensitivity analysis on a design-variable-random-distributed structure: fracture pattern at  $t_{\max} = 80 \mu\text{s}$  in (a) the half MBB-beam; (b) the cantilever beam; error of normalized sensitivity values (c) the half MBB-beam; (d) the cantilever beam; sensitivity values in elements at  $t_{\max} = 80$  in (e) the half MBB-beam and (f) the cantilever beam.

311 Fig. 8 depicts the sensitivity validation on a structure with random distribution of densities  
 312  $\varphi_e \in [0, 1]$  in the elements. After generating the densities using a uniform probability of distribu-  
 313 tions, a filter is then applied. The other parameters are provided in Table 2.

314 We can note that even though the absolute values of sensitivities are good, the relative errors  
 315 might locally be high, even though localized, associated with the approximation made in Eq. (53).  
 316 However, these errors remain acceptable. In addition, it will be shown in the next examples that  
 317 the made approximation allows a large simplification of the whole methodology, while keeping  
 318 important dynamic fracture reduction results.

### 319 5.2. Half-MBB beam

320 In this example, the presented methodology is applied to the Half-MBB beam (see Fig. 4(a)) to  
 321 minimize the fracture energy with respect to the topology of the structure. Three different values  
 322 of loading rates are investigated,  $v_0 = 20 \text{ m/s}$ ,  $v_0 = 40 \text{ m/s}$  and  $v_0 = 60 \text{ m/s}$ , respectively. Two  
 323 maximal loading times,  $t_{\max} = 40 \mu\text{s}$  and  $t_{\max} = 100 \mu\text{s}$ , are separately considered. The compliance

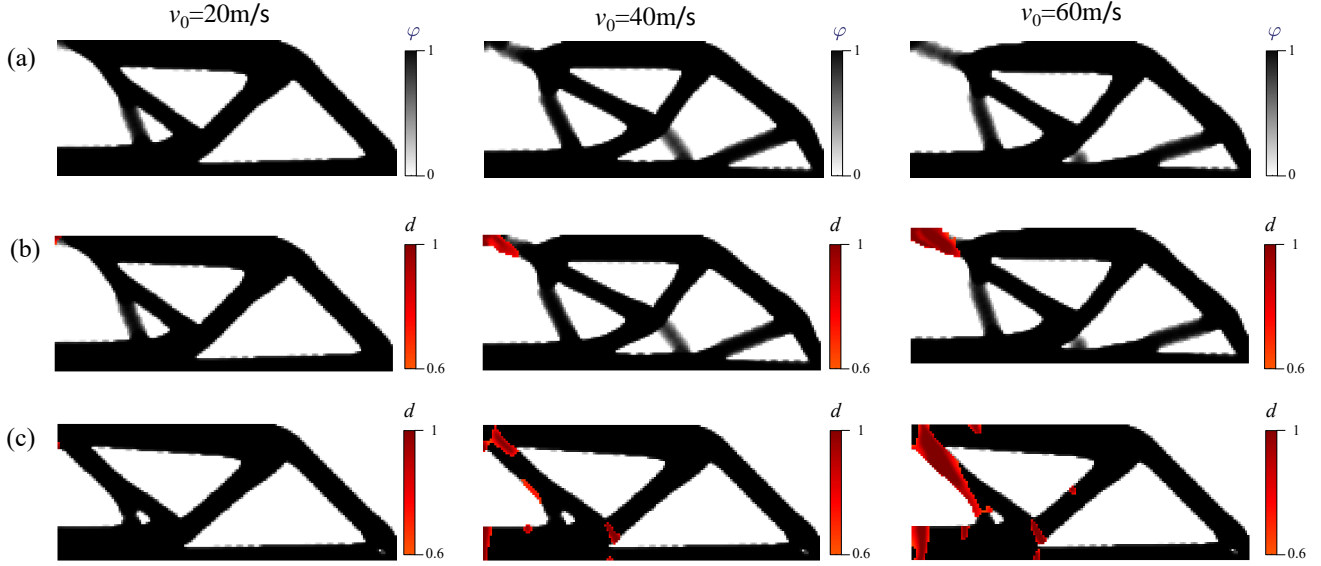


Figure 9: Obtained topological designs for  $t_{\max} = 40\mu\text{s}$  and corresponding final fracture patterns for different loading rates: (a) DF-design: final topology; (b) DF-design: final crack pattern; (c) S-design: final crack patterns.

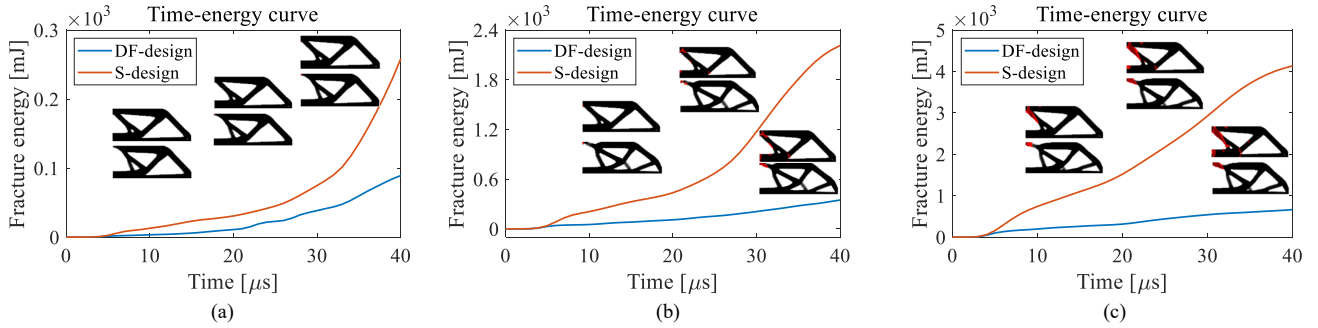


Figure 10: Comparison of time-energy curves for S- and DF-designs for  $t_{\max} = 40\mu\text{s}$  at different loading rates: (a)  $v_0 = 20\text{ m/s}$ ; (b)  $v_0 = 40\text{ m/s}$ ; (c)  $v_0 = 60\text{ m/s}$

324 constraints are defined by  $C_0 = 200\text{ N.mm}$  and  $C_0 = 300\text{ N.mm}$  for both loading times. The  
 325 volume fraction constraint is  $\chi = 0.5$ . The filter radius is chosen as  $r_{\min} = 4.5$  and  $\ell = 2\text{ mm}$ ,  
 326  $\Delta t = 1\mu\text{s}$ ,  $t_0 = 2\mu\text{s}$  and  $\text{tol} = 10^{-5}$ .

327 Fig. 9(a) shows the topological designs for  $t_{\max} = 40\mu\text{s}$  for the different loading rates. Material  
 328 on the right side of the loading area is removed because it is the location of cracks initiation  
 329 under high-speed impact. Fig. 9(b) shows the crack patterns at  $t_{\max} = 40\mu\text{s}$ . For comparison,  
 330 the crack patterns of the reference S-design (static case) are shown accordingly in Fig. 9(c).  
 331 Fig. 10 compares the fracture energy evolution in time for the present Dynamic Fracture DF-  
 332 and S- optimized designs, in a period of time  $[0-40\mu\text{s}]$ . An important decrease of the fracture  
 333 energy using the DF-design for all loading rates is appreciated, which shows the importance of

Table 3: Comparison of fracture energy using DF-design for  $t_{\max} = 40 \mu s$  and S-design at final simulation time for different loading rates.

Loading rate (m/s)	S-design	DF-design	Fracture energy reduction
20	258.23	91.51	64.5%
40	2213.33	347.92	84.3%
60	4133.94	627.92	84.8%

Table 4: Comparison on the fracture energy for DF-designs for  $t_{\max} = 100 \mu s$  and S-design at the final time for different loading rates.

Loading rate (m/s)	S-design	DF-design	Fracture energy reduction
20	2312.37	313.97	86.4%
40	5396.64	1007.77	81.2%
60	8127.18	2530.02	68.9%

334 including the dynamics in the topology optimization analysis as compared to the designs obtained  
 335 by simple static analysis. Corresponding comparisons and fracture energy reduction at  $t = 40 \mu s$   
 336 are presented in Table 3. When the loading rate increases, and thus the related dynamic effects,  
 337 the reduction of the fracture energy as compared to the one obtained by static analysis is even  
 338 larger.

339 Fig. 11 depicts the iterative process plots of the above topology optimizations. Regardless of  
 340 the jumps caused by the variation of the projection parameter  $\eta$ , a good convergence is appreciated.  
 341 All the optimization constraints are verified, except the compliance constraint for  $v_0 = 60 \text{ m/s}$ ,  
 342 which might be too strict to be reached in this case. The competition between minimizing  
 343 the fracture energy and satisfying the compliance constraint might be one possible reason for the  
 344 observed oscillations. For the case  $v_0 = 20 \text{ m/s}$ , 647 iterations were necessary, for a total of 8.7 h  
 345 on a single processor for the whole optimization process.

346 Next, a longer loading period is investigated, with  $t_{\max} = 100 \mu s$ . Fig. 12 depicts the topological  
 347 designs and their final fracture patterns under different loading rates. Compared to the designs for  
 348  $t_{\max} = 40 \mu s$  shown in Fig. 9, the obtained designs show an obvious difference, and the final fracture  
 349 patterns also change accordingly. Fig. 13 depicts the iterative processes of these topological  
 350 designs. In this case, although the objective function remains oscillatory, these oscillations remain



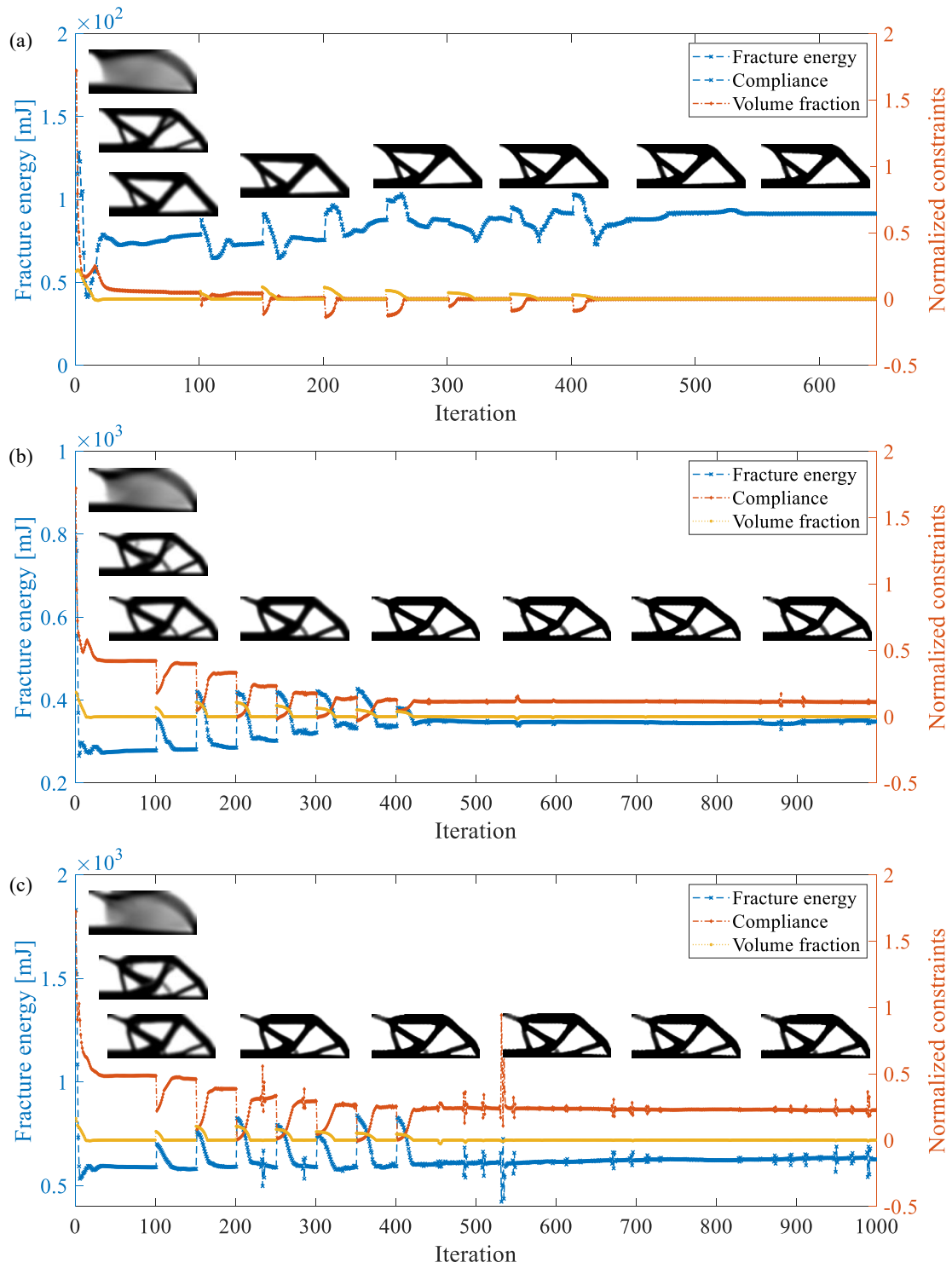


Figure 11: Iterative topology optimization process for  $t_{\max} = 40 \mu\text{s}$  under different loading rates: (a)  $v_0 = 20 \text{ m/s}$ ; (b)  $v_0 = 40 \text{ m/s}$ ; (c)  $v_0 = 60 \text{ m/s}$ .

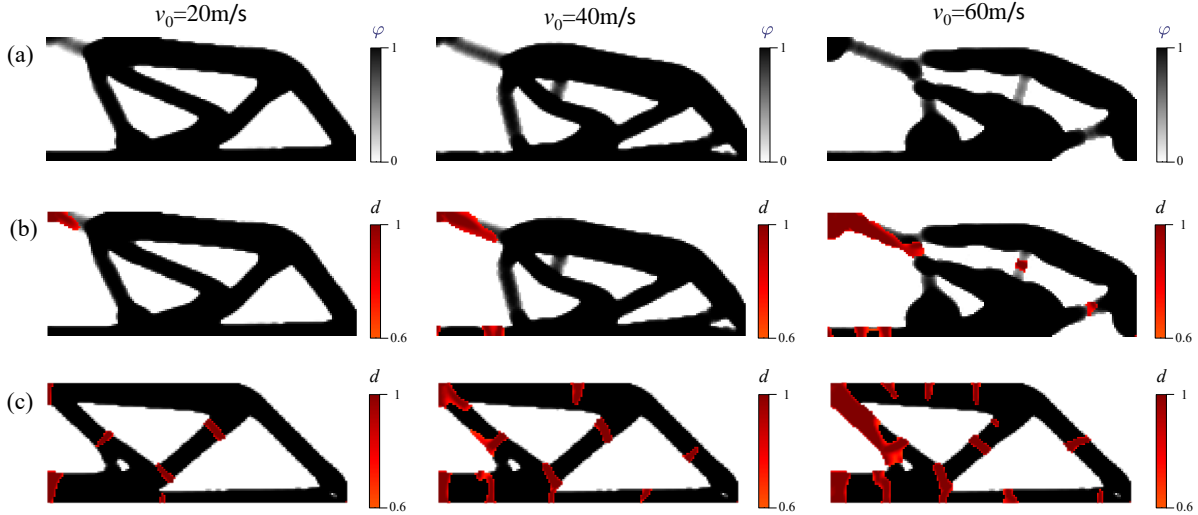


Figure 12: Topological designs for  $t_{\max} = 100 \mu s$  and corresponding final fracture patterns for different loading rates: (a) DF-design: final topology; (b) DF-design: final crack pattern; (c) S-design: final crack patterns.

351 small and around a stable value. The computational time for the case  $v_0 = 20\text{m/s}$  is 23 h for  
 352 672 iterations. The computational times are here proportional to the chosen loading period. Fig.  
 353 14 compares the fracture energy evolution in a period of time  $[0-100 \mu s]$  obtained by the present  
 354 DF-designs for  $t_{\max} = 100 \mu s$  and the S-designs. Once again, an important decrease of the fracture  
 355 energy using the DF-design for all loading rates is appreciated. Corresponding comparisons and  
 356 fracture energy reductions at  $t = 100 \mu s$  are indicated in Table 4.

### 357 5.3. Cantilever beam

358 In this section, the cantilever cantilever beam shown in Fig. 4(b) is investigated. Similarly,  
 359 three different values of loading rates,  $v_0 = 20 \text{ m/s}$ ,  $v_0 = 40 \text{ m/s}$  and  $v_0 = 60 \text{ m/s}$  are studied.  
 360 The maximal loading time in this example is defined by  $t_{\max} = 60 \mu s$ . The compliance constraint  
 361 is defined by  $C_0 = 200 \text{ N} \cdot \text{mm}$ . The volume fraction constraint is  $\chi = 0.5$ . The filter radius is  
 362 chosen as  $r_{\min} = 4.5$  and  $\ell = 2 \text{ mm}$ ,  $\Delta t = 1 \mu s$ ,  $t_0 = 2 \mu s$  and  $tol = 10^{-5}$ .

363 Fig. 15(a) shows the topological designs of the cantilever beam for different loading rates. Here,  
 364 the proposed method gives different topology designs for different loading rates. Fig. 15(b) shows  
 365 the crack pattern at  $t_{\max} = 60 \mu s$ . Fig. 15(c) depicts the final crack pattern of the S- design for  
 366 comparison. It is worth noting that there remain some gray elements in the DF- designs. This issue  
 367 is a classical one found by several other authors in dynamic topology optimization. For example, it  
 368 is discussed as a key issue in [58] and found in other works such as in [72, 73]. As the main objective  
 369 of this paper is to present the new topology optimization algorithm with fracture minimization

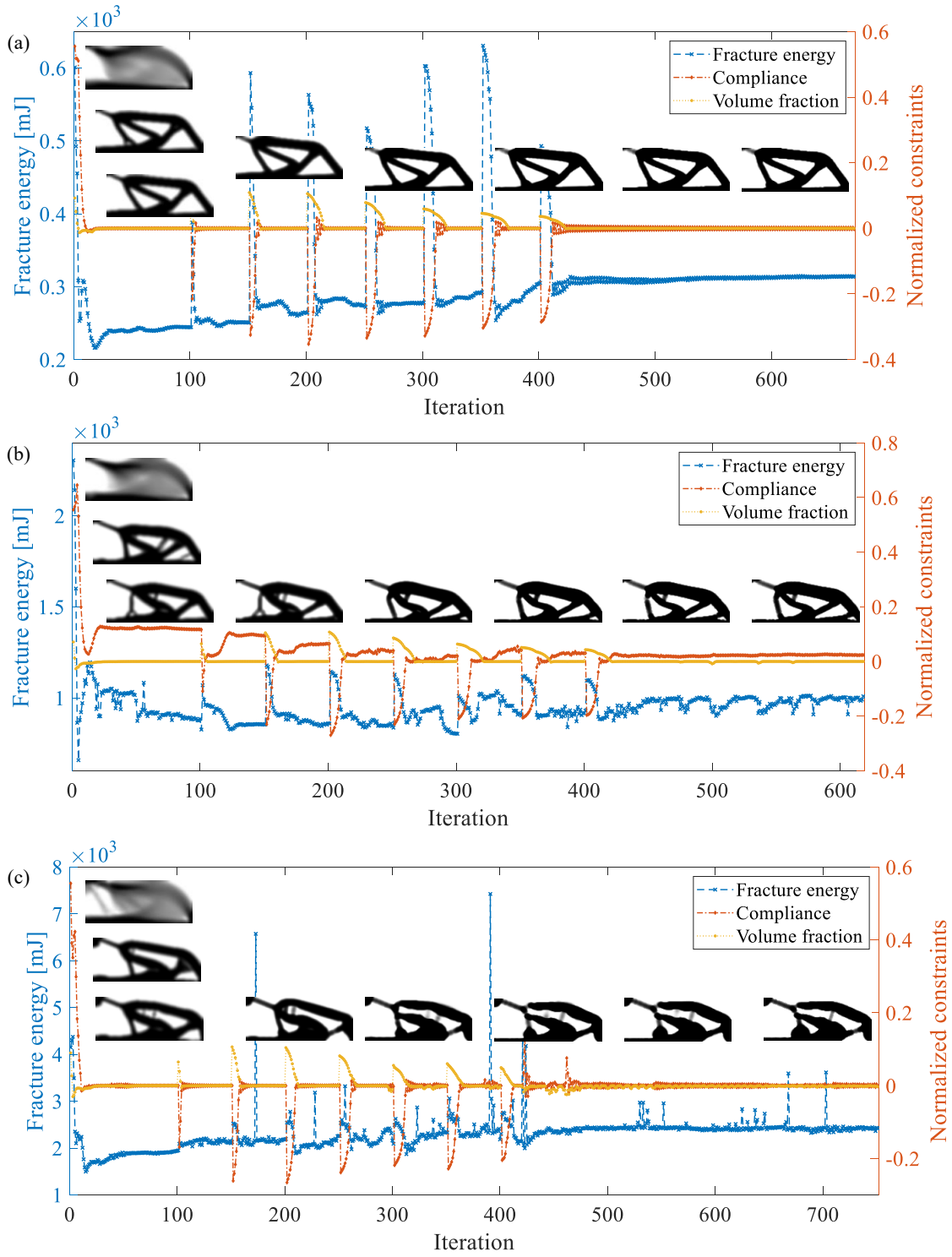


Figure 13: Iteration process of the topology optimization for  $t_{\max} = 100 \mu s$  for different loading rates: (a)  $v_0 = 20$  m/s; (b)  $v_0 = 40$  m/s; (c)  $v_0 = 60$  m/s.

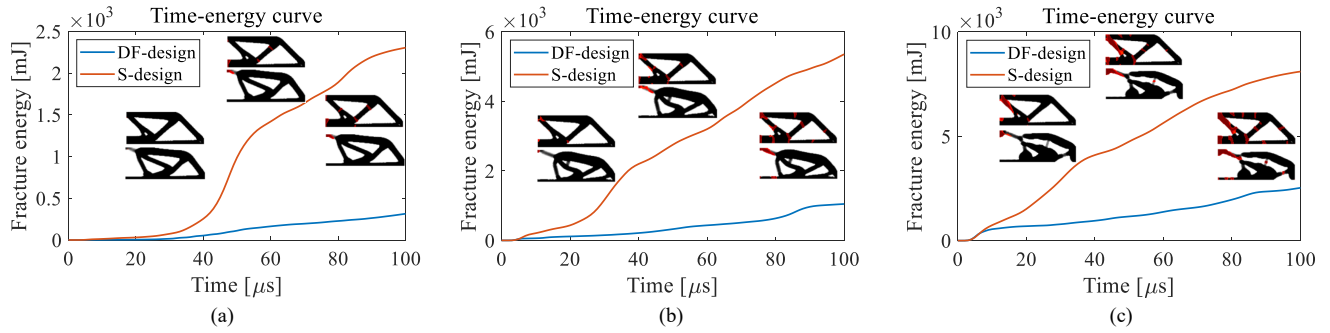


Figure 14: Comparison of time-energy curves for S- and DF-designs for  $t_{\max} = 100\mu s$  at different loading rates: (a)  $v_0 = 20$  m/s; (b)  $v_0 = 40$  m/s; (c)  $v_0 = 60$  m/s.

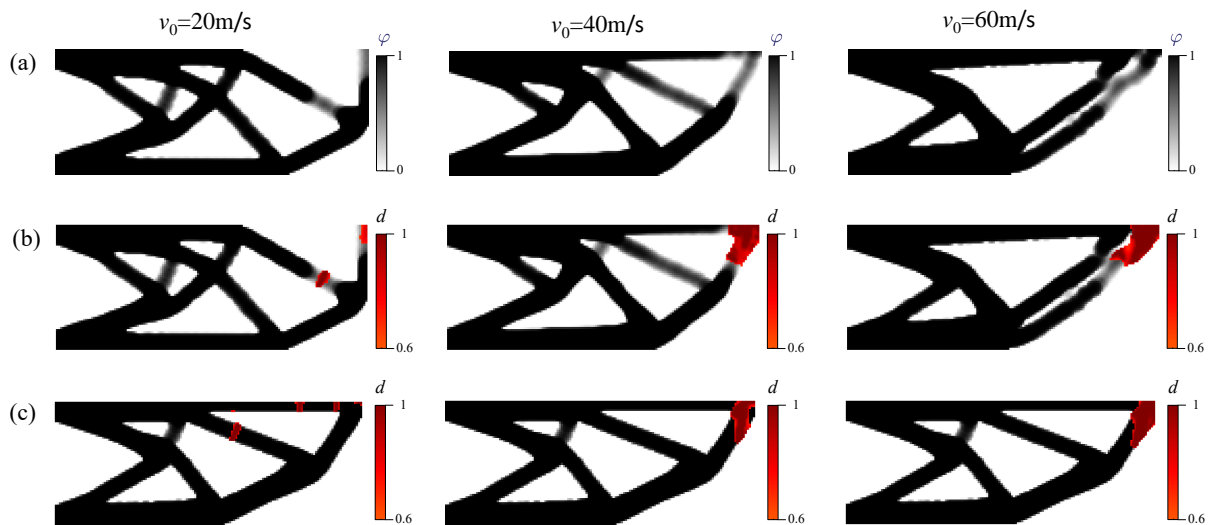


Figure 15: Topological designs for the cantilever beam and corresponding final fracture patterns for different loading rates: (a) DF-design: final topology; (b) DF-design: final crack pattern; (c) S-design: final crack patterns.

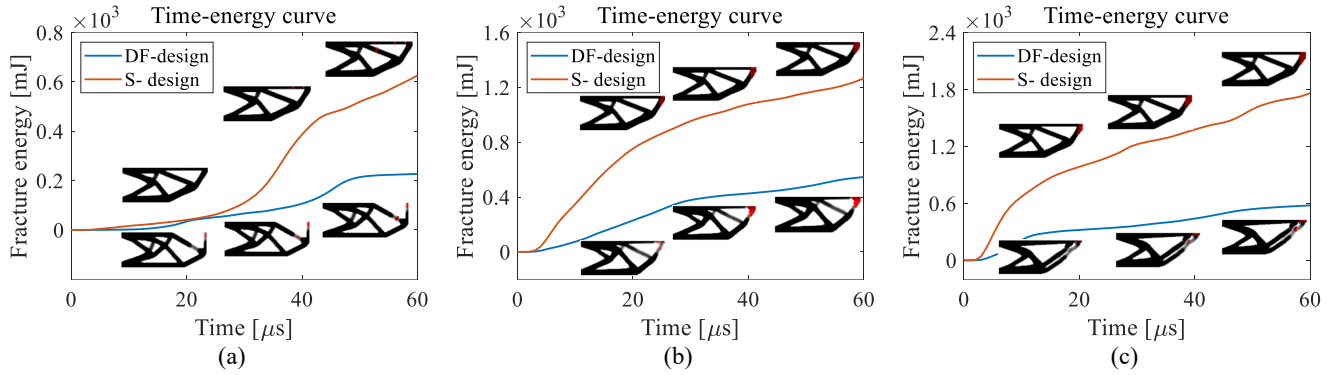


Figure 16: Comparison of time-energy curves for S- and DF-designs at different loading rates: (a)  $v_0 = 20$  m/s; (b)  $v_0 = 40$  m/s;  $v_0 = 60$  m/s.

Table 5: Comparison on the fracture energy of DF- and S-design for the cantilever beam at the final time for different loading rates.

Loading rate (m/s)	S-design	DF-design	Fracture energy reduction
20	638.56	227.20	64.4%
40	1289.30	546.75	57.6%
60	1788.05	578.54	67.6%

370 objective, fully addressing this problem is reported to later studies. Fig. 16 depicts the fracture  
371 energy evolution of the DF- and S- designs in a period of time  $[0-60 \mu\text{s}]$ . Table 5 provides the  
372 comparison of fracture energy for different loading rates of the DF- and S- designs at the final  
373 time. Again, the DF-designs show large reductions of the fracture energy. Further investigations,  
374 including comparisons with stress-based linear topology optimization, could be conducted in future  
375 studies.

376 Fig. 17 depicts the iterative processes of the topology optimizations for different loading rates.  
377 A good convergence is obtained and all the constraints are reached. The computational time for  
378 the case  $v_0 = 20\text{m/s}$  is 12.7 h for 624 iterations.

379 As a final remark, we can note that in most studied examples, the cracks are rather diffuse  
380 damage zones. In the present phase field framework, the cracks width depends on the mesh  
381 density. To maintain reasonable computational costs, we used meshes which do not allow very fine  
382 descriptions of cracks. However, it has been shown in many other studies (see e.g. [45]) that the  
383 phase field method is fully convergent with respect to the mesh density, even in the dynamic case.  
384 Then, finer crack descriptions can be obtained if faster computational resources are available.

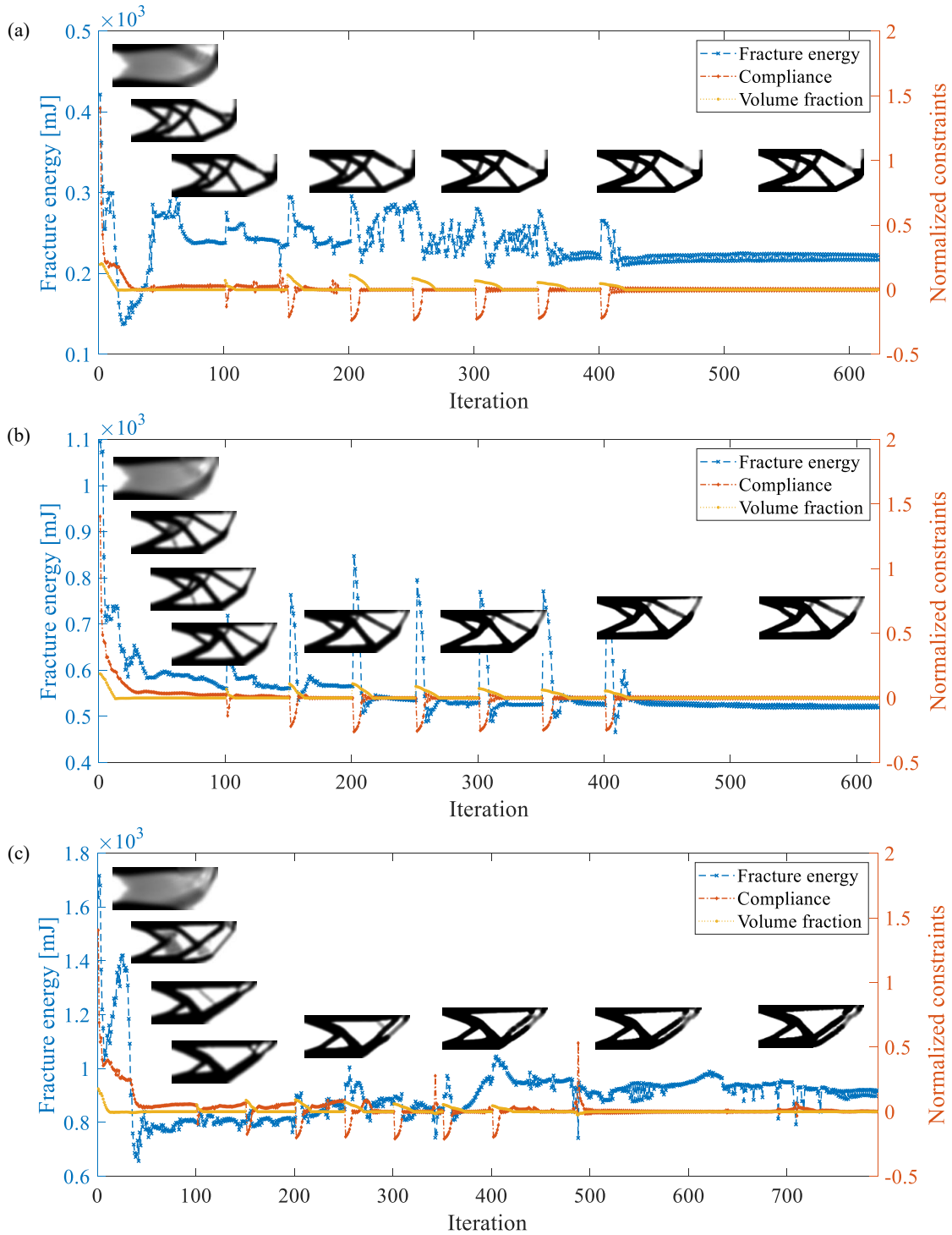


Figure 17: Iteration process of the topology optimization of cantilever beam for different loading rates: (a)  $v_0 = 20$  m/s; (b)  $v_0 = 40$  m/s; (c)  $v_0 = 60$  m/s.

## 385 **6. Conclusion**

386 A SIMP topology optimization framework for maximizing the dynamic fracture resistance has  
387 been proposed. Several contributions have been introduced. The dynamic phase field method for  
388 fracture has been combined with SIMP topology optimization. Then, a topology optimization  
389 minimizing the fracture optimization as an objective function under constraints of material vol-  
390 ume and verification of local equilibrium equations has been originally proposed. Semi-analytical  
391 expressions of sensitivities in this context have been derived, and their accuracy using numerical  
392 finite difference approximations has been validated. The algorithm involves solving at each iter-  
393 ation first a static problem to evaluate the compliance and then a full dynamic fracture problem  
394 from initiation to crack propagation, during a given period of time, then taking into account the  
395 whole loading history. A staggered scheme with convergence iterations has been used to solve the  
396 dynamic phase field problem thus authorizing larger time steps. Numerical examples on structural  
397 problems subjected to impacts for different loading velocities have been investigated. The exam-  
398 ples show that the present dynamic analysis allows reducing the fracture energy as compared to  
399 the designs obtained from static classical topology optimization analysis.

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